Problem 1

a) $100n + \log n = \Theta(n+(\log n)^2)$

The n term dominates here, and at a large enough n the log n terms are not relevant.

b) $\log n = \Theta(\log(n^2))$

$\log(n^2) = 2 \log n$ by the properties of logs

So $\log n <= 2 \log n$ and vice versa, so this relation is theta.

c) $n^2/\log n = \Omega(n (\log n)^2)$

$n^2/\log n >= c(n (\log n)^2)$

$n^2 >= c(n (\log n)^3)$

$n*n>=cn(\log n)^3$

Clearly, $(\log n)^3$ grows more slowly than n.

However, $n^2/\log n \neq \Theta(n (\log n)^2)$

d) $(\log n)^{\log n} = \Omega(n/\log n)$

$(\log n)^{\log n} >c\ 2^{\log n}$ for any $n>4$

$2^{\log n} = n$

$\log n^{\log n} > 2^{\log n}= n>n/\log n$

However, $(\log n)^{\log n} \neq \Theta(n/\log n)$

e) $n^{1/2} = \Omega((\log n)^5)$

$n^{1/2} >= c (\log n)^5$ //square both sides

$n >= c(\log n)^{10}$

This is true for sufficiently large n.

However, $n^{1/2} \neq \Theta((\log n)^5)$

f) $n 2^n = O(3^n)$

$n 2^n <= c 3^n$

$n <= c 3^n / 2^n$

$n<=c(1.5)^n$

This is true as exponential functions increase faster than polynomials.

However, $n 2^n \neq \Theta (3^n)$

Problem 2

T(n) = T(n-2) + T(2) + a*n
T(n) = b if n<=2 , T(2) = b
T(n) = T(n-2) + a*n + b

T(n) = (T(n-4) +a(n-2)+b) + a*n + b
= T(n-4) + a*(n-2) + a*n + 2b
=(T(n-6)+a*(n-4)+b) + a*(n-2) + a*n + 2b
= T(n-6) + a*(n-4)+a*(n-2)+a*n +3b

$$=T(n-2i) + \sum_{j=0}^{i-1} (a*n-2j) + i*b$$

This will stop when i = (n/2-1), so substituting and simplifying:

$$= T(n-2(n/2-1)) + \sum_{j=0}^{(n/2-1)-1} (a*n-2j) + (n/2-1)*b$$

$$= T(2) + \sum_{j=0}^{n/2-2} (a*n-2j)+ (n/2-1)*b$$

$$= b + \sum_{j=0}^{n/2-2} (a*n-2j) + (n/2-1)*b$$

$$= \sum_{j=0}^{n/2-2} (a*n-2j) + (n/2)*b$$

$$= a* \sum_{j=0}^{n/2-2} (n-2j) + (n/2)*b$$

$$= a* \sum_{j=0}^{n/2-2} n - 2a* \sum_{j=0}^{n/2-2} j + (n/2)*b$$

$$= a *n*(n/2-1) - 2a * \sum_{j=0}^{n/2-2} j + (n/2)*b$$

=a*n*(n/2-1) - 2a* (n/2-2+1)*(n/2-2)/2+(n/2)*b
= a*n*(n/2-1) - a* (n/2-1)*(n/2-2)+(n/2)*b
Expanding this polynomial gives:

$$= \frac{an^2}{4} + \frac{bn}{2} + \frac{an}{2} - 2a$$

Thus, T(n) = O(n^2)

Problem 3

Prove $T(n) = 2T(n-c)+k$

$=2(2T(n-2c)+k)+k$

$=2^2T(n-2c)+(2+1)k$

$=2^2\ (2*T(n-3c)+k)+(2+1)k$

$=2^3T(n-3c)+2^2k+(2+1)k$

$=2^3T(n-3c)+\ (2^2+2+1)k$

$=2^iT(n-ic)+\ \sum_{j=0}^{i-1}\left(k2^j\right)$

$=2^iT(n-ic)+\ k*\sum_{j=0}^{i-1}\ (2^j)$

This ends with i=n/c

$=2^{n/c}T(0)+k*\ \sum_{j=0}^{n/c-1}\ (2^j)$

Notice that the summation is a geometric series, and the sum of a geometric series is known is known, so by the substitution on page 53 of Manber. Also, assuming that T(0) is a constant.

$=2^{n/c}c_2+k*\ \sum_{j=0}^{n/c-1}\ (2^j)$

$=2^{n/c}c_2+k*(2^{n/c}-1)$

$=(k+c_2)*2^{n/c}-k$

$=O(2^{n/c}) = O((2^{1/c})^n)$ for sufficiently large n.

This proves the relationship desired, where $d = 2^{1/c}$

Problem 4

Note that according to the problem, the combination step takes at most $a*n^2$ time.

This gives $T(n) = 3T(n/3) + a*n^2$

$= 3(3T(n/3^2)+a(n/3)^2)+a*n^2$

$=3^2T(n/3^2) + 3*a*(n/3)^2+a*n^2$

$=3^2(3T(n/3^3)+a*(n/3^2)^2)+ 3*a*(n/3)^2+a*n^2$

$=3^3T(n/3^3)+3^2 a*(n/3^2)^2 + 3*a*(n/3)^2+a*n^2$

$=3^iT(n/3^i) + \sum_{j=0}^{i-1} (3^j a*(n/3^j)^2)$

$=3^i T(n/3^i) + \sum_{j=0}^{i-1} (3^j /3^{2j} a*n^2)$

$=3^i T(n/3^i) + a*n^2 * \sum_{j=0}^{i-1} (3^j /3^{2j})$

$=3^i T(n/3^i) + a*n^2 * \sum_{j=0}^{i-1} (1/3^j)$

Stops at $i = \log_3 n$

$=3^{\log3 n} T(n/3^{\log3 n}) + a*n^2 * \sum_{j=0}^{i-1} (1/3^j)$

$=n T(n/n) + a*n^2 * \sum_{j=0}^{i-1} (1/3^j)$

$=n T(1) + a*n^2 * \sum_{j=0}^{i-1} (1/3^j)$

$=c*n + a*n^2 * \sum_{j=0}^{i-1} (1/3^j)$

$=c*n + a*n^2 * \sum_{j=0}^{i-1} (3^{-j})$

$=c*n + a*n^2 *(\frac{3}{2}(1-3^{-i}))$

$=c*n + \frac{3}{2}a*n^2 - \frac{3}{2}(3^{-i})$

$=c*n + \frac{3}{2}a*n^2 - \frac{3}{2}*a*n^2*1/(3^{\log3 n})$

$=c*n + \frac{3}{2}a*n^2 - \frac{3}{2}*a*n^2*1/n$

$=\frac{3}{2}a*n^2+(c-\frac{3}{2}a)*n$

Thus, $T(n) = O(n^2)$